5 Scalable and Hilbert spaces

Definition 5.1 *Let X be a vector space over $\mathbb{K}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called* **scalar product** *iff, for all $x, y, z \in X$ and all $k \in \mathbb{K}$*

1. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
2. $\langle kx, y\rangle = k\langle x, y\rangle$,
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
4. $\langle x, x\rangle \geq 0$,
5. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

We conclude immediately that we have as well

1. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,
2. $\langle x, ky\rangle = \overline{k}\langle x, y\rangle$.

Exercise: In the Brazilean jungle there is an armed group of rebellious mathematicians claiming that the 'real, eternal and unique definition of a scalar product is the one of a nondegenerate symmetric bilinear form with $\langle x, x\rangle \in (-\infty, 0] \cdot i$'. Who's right?

Theorem 5.2 (Cauchy-Schwarz inequality) *Let $\langle \cdot, \cdot \rangle$ be a scalar product on a vector space V. Then we have, for all $x, y \in V$,*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle.$$

If $\{x, y\}$ is linearly independent, then the inequality is strict.

Proof. For any $s \in K$, we have

$$\begin{aligned} 0 &\leq \langle x + sy, x + sy\rangle \\ &= \langle x, x\rangle + s\overline{\langle x, y\rangle} + \overline{s}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle. \end{aligned}$$

With the special choice $y \neq 0, s := -\frac{\langle x,y\rangle}{\langle y,y\rangle}$ we get

$$0 \leq \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} + \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}$$

which gives immediately the Cauchy-Schwarz inequality. Equality implies $x + sy = 0$. □

Theorem 5.3 *Let V be a vector space with a scalar product $\langle \cdot, \cdot \rangle$. Then $||x|| := \sqrt{\langle x, x\rangle}$ defines a norm.*

Proof. We have to show sublinearity (triangle inequality at 0):

$$||x + y||^2 = ||x||^2 + 2Re\langle x, y\rangle + ||y||^2 \leq ||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 = (||x|| + ||y||)^2,$$

where the inequality uses the Cauchy-Schwarz inequality and the definition of absolute value in $\mathbb{K}$. □

Theorem 5.4 *A norm $||\cdot||$ in a tvs X is associated to a scalar product as in Theorem 5.3 if and only if, for all $x, y \in X$,*

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2. \tag{1}$$

Moreover, in this case, the scalar product is continuous.

Proof. We treat the real case leaving the complex one as an **exercise**. The fact that, if the norm is defined by a scalar product, the equation holds, is quite trivial. If Equation 1 holds, then the *only* map from $X \times X$ to $\mathbb{R}$ inducing the norm as in Theorem 5.3 is

$$\langle x, y\rangle := \frac{1}{4}(||x+y||^2 - ||x-y||^2).$$

(prove that, **exercise**). It is easy to check that then indeed $||x||^2 = \langle x, x\rangle$ and that $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ is continuous. We will use Equation 1 to prove that $\langle\cdot,\cdot\rangle$ is a scalar product. We will only show linearity in the first argument, the other properties being easy to seen as direct consequences.
Now, for $x, y, z \in X$, Equation 1 implies

$$\begin{aligned} ||x+y+z||^2 &= 2||x+z||^2 + 2||y||^2 - ||x-y+z||^2 =: a, && (2)\\ ||x+y+z||^2 &= 2||y+z||^2 + 2||x||^2 - ||-x+y+z||^2 =: b. && (3)\end{aligned}$$

Therefore we have

$$\begin{aligned} ||x+y+z||^2 &= \frac{a+b}{2} \\ &= ||x+z||^2 + ||x||^2 + ||y+z||^2 + ||y||^2 \\ &- \frac{1}{2}(||x-y+z||^2 + ||-x+y+z||^2) \end{aligned}$$

and analogously

$$\begin{aligned} ||x+y-z||^2 &= ||x-z||^2 + ||x||^2 + ||y-z||^2 + ||y||^2 \\ &- \frac{1}{2}(||x-y-z||^2 + ||-x+y-z||^2). \end{aligned}$$

This implies

$$\begin{aligned} \langle x+y, z\rangle &= \frac{1}{4}(||x+y+z||^2 - ||x+y-z||^2) \\ &= \frac{1}{4}(||x+z|^2 + ||y+z||^2 - ||x-z||^2 - ||y-z||^2) \\ &= \langle x, z\rangle + \langle y, z\rangle. \end{aligned}$$

For the scalar multiplication, observe that by the above we already have $\langle sx, y\rangle = s\langle x, y\rangle$ for $s \in \mathbb{N}$ and by the usual easy argument also for $s \in \mathbb{Z}$. For rational $s = \frac{m}{n}$ observe

$$n\langle sx, y\rangle = n\langle m \cdot (\frac{1}{n} \cdot x), y\rangle = m\langle x, y\rangle = ns\langle x, y\rangle$$

which implies the statement for s rational. Now $s \to s\langle x, y\rangle - \langle sx, y\rangle$ is a continuous real function on $\mathbb{R}$ and vanishes on the rationals, therefore it vanishes everywhere. $\square$

Exercise: Which of the norms seen so far come from a Hilbert product in the sense of Theorem 5.3? In particular, what about $B(X, Y)$ for X normed, Y Banach?

As an easy corollary, we get

Theorem 5.5 *A normed space has an associated scalar product if and only if all its two-dimensional subspaces are Euclidean. Linear subspaces of scaled spaces are scaled spaces. The completion of a scaled space is a Hilbert space.*

Proof: Exercise. $\square$

Exercise: Let S be any set. Define $A(S)$ as the set of almost everywhere vanishing $\mathbb{K}$-valued functions (w.r.t. the counting measure, i.e. everywhere except in countably many points). Now for $f \in A(S)$ define $||f|| := \sum |f(x)|^2$. Show that $|\cdot|$ is a well-defdined map taking values in $[0, \infty]$. Now define $l^2(S) := \{f \in A(S) : ||f|| < \infty\}$. Show that this is a Hilbert space.
Remark for Connaisseurs: We could have defined $l^2(S) := L^2(S, \tau, \mu_c)$ for τ the discrete topology and μ the counting measure.
Remark: This kind of Hilbert spaces will be important once we introduced orthonormal bases.

Definition 5.6 *Let $X, \langle \cdot, \cdot \rangle$ be a scaled space. Two vectors x, y are called* **orthogonal***, $x \perp y$, iff $\langle x, y\rangle$. Two subsets A, B of X are called* **orthogonal***, $A \perp B$, if $x \perp y$ whenever $x \in A, y \in B$. Given a subset A of X, the subset of all vectors w of V with $\{w\}$ orthogonal to V is called* **orthogonal complement of** V *and denoted by $V^\perp$.*

We see immediately that

Theorem 5.7 *Given a subset A of a scaled space X, we have that $A^\perp = (\overline{\langle A \rangle})^\perp$ and that $A^\perp$ is a closed linear subspace of X. Moreover, $A \subset (A^\perp)^\perp$. In general, the converse of the last inclusion is wrong.*

Proof: Exercise. $\square$

Theorem 5.8 *Let H be a Hilbert space, let $C \subset H$ be closed and convex. Then there is a continuous and even contractive projection $P_K : H \to K$ with $||P_K(h) - h|| = dist(h, K)$. We have $Re\langle x - P_K(x), y - P_K(x)\rangle \leq 0$ for all $y \in K$.*

Proof: Exercise. Hint: considering the theorem in the Banach section You will have to prove that Hilbert spaces are strictly convex and that the minimizing sequence converges. $\square$

The direct sum $H_1 \oplus H_2$ of two scaled spaces carries the **standard direct sum scalar product** $\langle (v_1, v_2), w_1, w_2)\rangle := \langle v_1, w_1\rangle_1 + \langle v_2, w_2\rangle_2$.

Theorem 5.9 *Let H be a Hilbert space and C be a closed subspace. Then there is exactly one surjective continuous linear map $P_C : H \to C$ with the following properties:*

1. *P is a projection, $P_C^2 = P_C$,*
2. *$ker(P_C) = C^\perp$*

Automatically we have $||P_C|| = 1$. We call P_C the **orthogonal projection to** *C. H is isometric to $C \oplus C^\perp$ with the stanard direct sum scalar product. Defining the orthogonal projection for all closed proper subspaces, we get the relation $1 - P_C = P_{C^\perp}$.*

Proof. Obviously every linear subspace of H is convex, thus by the Theorem 5.8 we get a continuous projection P_C with

$$Re\langle x - P_C(x), y - P_C(x)\rangle \leq 0 \qquad \forall y \in U, x \in H. \tag{4}$$

Setting $z := y - P_C(x) \in C$ we get

$$Re\langle x - P_C(x), z\rangle \leq 0 \qquad \forall z \in U, x \in H$$

and, considering that $i \cdot y, -y \in C$, we get

$$\langle x - P_C(x), z\rangle = 0 \qquad \forall z \in U, x \in H,$$

and conversely from this follows Equation 4. Therefore $P_C(x)$ is given uniquely by the condition

$$x - P_C(x) \in C^\perp. \tag{5}$$

This characterization implies trivially linearity and the kernel property. The last property is easy to see by turning around the argument: By Theorem 5.7 we know that $C^\perp$ is a closed subspace. Now by 5, $\mathbf{1} - P_C$ takes values in $C^\perp$, and $x - (\mathbf{1} - P_C)(x) = P_C(x) \in P \subset (C^\perp)^\perp$. Therefore $(\mathbf{1} - P_C)$ satisfies the defining condition 5 for $C^\perp$. □

As a corollary, we get

Theorem 5.10 *For every linear subspace U of a Hilbert space H we have $\overline{U} = (U^\perp)^\perp$.*

Proof. Put $V := \overline{U}$. By continuity, $V^\perp = U^\perp$, and by the preceeding theorem we have $P_{(V^\perp)^\perp} = \mathbf{1} - P_{V^\perp}$. Therefore $P_{\overline{U}} = P_V = P_{(V^\perp)^\perp} = P_{(U^\perp)^\perp}$ which implies the statement of the theorem. □

Theorem 5.11 (Frechet-Riesz Representation Theorem) *Let H be a Hilbert space. Then the map $\phi : v \mapsto \langle v, \cdot\rangle$ is a linear isometric isomorphism between H and H^*.*

Proof. Linearity is trivial from sesquilinearity of the scalar product. The Cauchy-Schwartz inequality implies $||\phi(v)||_{H^*} \leq ||v||_H$, and for $v \neq 0$ we get $(\phi(v))(||v||^{-1}v) = ||v||$, therefore ϕ is an isometry and therefore injective. It remains to show that ϕ is surjective. Now let a nonzero vector $y \in H^*$ be given, then define $K := kery \subset H$ which is a closed linear subspace. Following Theorem 5.9, K has an orthogonal complement C on which moreover y is injective, so C is one-dimensional and can be written as $C := \mathbb{K} \cdot x$ where x is a normalized vector in H, so H is isometric to $K \oplus \mathbb{K} \cdot x$, and thus for $r := y(x)$ and $\tilde{x} := r \cdot x$, then using the direct sum decomposition it is easy to check that $w = \phi(\tilde{x})$, thus ϕ is also surjective. □

Theorem 5.12 (Bilinear maps) *Let H be a complex Hilbert space and $B : H \times H \to \mathbb{C}$ sesquilinear. Then the following statements are equivalent:*

1. *B is continuous.*

2. *$y \mapsto B(x,y)$ is continuous and $x \mapsto B(x,y)$ are continuous for all $x, y \in H$.*

3. *There is an $M \geq 0$ with $|B(x,y)| \leq M||x|| \cdot ||y||$ for all $x, y \in H$.*

If B is continuous, there is a continuous linear map $L : H \to H$ with $B(x,y) = \langle Lx, y \rangle$, for all $x, y \in H$. If there is additionally an $m > 0$ s.t. for all $x \in H$ we have $B(x,x) \geq m||x||^2$ (this property is called **coercivity***), then L is invertible with $||L^{-1}|| \leq m^{-1}$.*

Proof: Exercise. □

Definition 5.13 *Let V be a scaled space. A subset S of V is called* **orthonormal system** *iff $\langle s, s \rangle = 1$ for all $s \in S$ and $\langle s, t \rangle = 0$ for $s, t \in S, s \neq t$. An orthonormal system S is called* **maximal** *iff every orthonormal system containing S is equal to S.*

Example-Exercise: Give a maximal orthonormal system of $l^2(\mathbb{N})$!

Example: Let $I := [0, 2\pi]$. Then, for $x \in I$ define $s_0(x) := \frac{1}{\sqrt{2\pi}}$, and, for $n \in \mathbb{N}$, $s_{2n-1}(x) := \frac{1}{\sqrt{\pi}} sin(n \cdot x)$, $s_{2n}(x) := \frac{1}{\sqrt{\pi}} cos(n \cdot x)$. Then the set $\{s_n | n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal system (use partial integration!).

Example-Exercise: Show that $S = \{x \mapsto \frac{1}{\sqrt{2\pi}} e^{inx} | n \in \mathbb{Z}\}$ is an orthonormal system in $L^2(I, \mathbb{C})$!

Theorem 5.14 (Gram-Schmidt method) *Let S be a countable linear independent subset of a scaled space V. Then there is an orthonormal system N with $\overline{\langle N \rangle} = \overline{\langle S \rangle}$.*

Proof. Define $e_1 := \frac{S_1}{||S_1||}$ and inductively

$$e_{k+1} := \frac{S_{k+1} - \sum_{i=1}^{k} \langle S_{k+1}, e_i \rangle e_i}{||S_{k+1} - \sum_{i=1}^{k} \langle S_{k+1}, e_i \rangle e_i||}$$

which is well-defined as the enumerator is a non-zero vector because of linear independence. Then $E := \{e_n | n \in \mathbb{N}\}$ is an orthonormal system. By construction we have $e_n \in \langle S \rangle$, and inductively we get $S_n \in \langle E \rangle$ as well, thus the same holds for their closures. □

Theorem 5.15 (Bessel's inequality) *Let V be a scaled space and $\{e_n | n \in \mathbb{N}\}$ a orthonormal system in V, let $v \in V$ be arbitrary. Then we have*

$$\sum_{i=1}^{\infty} |\langle v, e_i \rangle| \leq ||v||^2.$$

Proof. Let, for abitrary $N \in \mathbb{N}$, $P_N v := \sum_{i=1}^{N} \langle v, e_i \rangle e_i$ be the orthogonal projection on $A_N := \langle \{e_1, ...e_n\} \rangle$, then $Q_N := \mathbf{1} - P_N$ is the orthogonal projection on $A_N^{\perp}$, and by $v = P_n v + Q_n v$ we get

$$\begin{aligned} ||v||^2 &= ||Q_n v||^2 + ||\sum_{n=1}^{N} \langle v, e_n \rangle e_n||^2 \\ &= ||Q_n v||^2 + \sum_{n=1}^{N} |\langle v, e_n \rangle|^2 \geq \sum_{n=1}^{N} |\langle v, e_n \rangle|^2. \end{aligned}$$

By taking the limit we get the statement. $\square$

As immediate corollaries we get

Theorem 5.16 *Let $\{e_n | n \in \mathbb{N}\}$ be an orthonormal system in a scaled space V, let $v, w \in V$. Then*

$$\sum_{i=1}^{\infty} |\langle v, e_i \rangle \langle e_i, w \rangle| \leq ||v|| \cdot ||w||.$$

Proof. Combine the Cauchy-Schwarz inequality in $l^2(\mathbb{N})$ with Bessel's inequality. $\square$

Theorem 5.17 *Let S be an orthonormal system in a scaled space V, let $v \in V$. Then $S_v := \{s \in S | \langle s, v \rangle \neq 0\}$ is at most countable.*

Proof. Define $S_{v,n} := \{s \in S | \langle s, v \rangle > \frac{1}{n}\}$. Bessel's inequality implies that every $S_{v,n}$ is finite. Therefore S_v as countable union of finite sets is countable. $\square$

With this in mind, and taking into account that the terms in the sum of Bessel's inequality are nonnegative, one can formulate and immediately prove the following

Theorem 5.18 (General Bessel's Inequality) *Let S be an orthonormal system in a scaled space V, then*

$$\sum_{s \in S} |\langle v, s \rangle|^2 \leq ||v||^2$$

for every $v \in V$. $\square$

Remark. Of course, the relevance of the more general version of the theorem lies in its applications to non-separable scaled spaces. There, we do not have a prescribed order in summing up the terms, thus we need the notion of **unconditional** convergence. A series converges unconditionally to a vector iff it converges to the same vector after any permutation of the terms.

Theorem 5.19 *Let V be a Hilbert space and E be an orthonormal system in V. Then:*

1. *For all $v \in V$, the series $\sum_{e \in E} \langle v, e \rangle e$ converges unconditionally.*
2. *The map $P_E : v \mapsto \sum_{e \in E} \langle v, e \rangle e$ is the orthogonal projection on $\overline{\langle E \rangle}$.*

Proof. To prove the first statement, let a bijective $S : \mathbb{N} \to E_v$ be an ordering of E_v (which is countable following Theorem 5.17) and $S \circ \pi$ another ordering. Bessel's inequality implies that for $N, M \to \infty$ we have

$$||\sum_{i=N}^{M} \langle v, e_i \rangle e_i|| = \sum_{i=N}^{M} |\langle v, e_i \rangle e_i|^2 \to 0$$

and analogously for the other ordering, thus both series are Cauchy and thus have limits

$$w := \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i, \qquad w_\pi := \sum_{i=1}^{\infty} \langle v, e_{\pi(i)} \rangle e_{\pi(i)}.$$

We have to show that $w = w_\pi$. Let $x \in V$ be arbitrary, then

$$\langle w, x\rangle = \sum_{i=1}^{\infty} \langle w, e_i\rangle \cdot \langle e_i, x\rangle = \sum_{i=1}^{\infty} \langle w, e_{\pi(i)}\rangle \cdot \langle e_{\pi(i)}, x\rangle = \langle w_\pi, x\rangle$$

(the second equality uses absolute and thus unconditional convergence following Theorem 5.16). For the second statement, considering Equation 5 Theorem 5.9, we have to show that

$$x - \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i, e\rangle = 0$$

for all $e \in E$. Either $\langle x, e\rangle = 0$, then this equation holds trivially. Or $\langle x, e\rangle \neq 0$, then following Theorem 5.17 we have that there is an $N \in \mathbb{N}$ with $e = e_N$, then the equation holds as well. □

Theorem 5.20 *Let S be an orthonormal system in a Hilbert space H. Then there is a maximal orthonormal system $\hat{S}$ containing S. Moreover, the following statements are equivalent:*

1. *S is maximal.*
2. *$S^\perp = \{0\}$.*
3. *$\overline{\langle S\rangle} = H$.*
4. *For all $x \in H$ we have $x = \sum_{s \in S} \langle x, s\rangle s$.*
5. *For all $x, y \in H$ we have $\langle x, y\rangle = \sum_{s \in S} \langle x, s\rangle\langle s, y\rangle$.*
6. *For all $x \in H$ we have $||x|| = \sum_{s \in S} |\langle x, s\rangle|^2$.*

Proof. The first property follows directly from Zorn's Lemma.
(1) ⇒ (2): If $S^\perp \ni v \neq 0$, then $S' := S \cup \{v/||v||\}$ is an orthonormal system properly containing S, thus S was not maximal.
(2) ⇒ (3): Use Theorem 5.10.
(3) ⇒ (4): As the closure of the linear span of S is the whole Hilbert space, for every $x \in H$ there is a sequence of coefficients $a_n \in \mathbb{K}$ and $s_n \in S$ with $\sum a_n s_n = x$. But the coefficients are given uniquely by the scalar products.
(4) ⇒ (5): Using (4), there are $s_n \in S, n \in \mathbb{N}$, and $x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$. Then use bilinearity and continuity of the scalar product and convergence of the series.
(5) ⇒ (6) Trivial.
(6) ⇒ (1) Suppose S is not maximal, choose an orthonormal system $\overline{S}$ properly containing S and a normalized element n of $\overline{S} \setminus S$. Then we get $1 = ||n|| = \sum_{s \in S} \langle n, s\rangle = 0$, a contradiction. □

As a corollary, we get (note that the coefficients are unique as scalar products):

Theorem 5.21 *A maximal orthonormal system is a Schauder basis (and therefore also called* **orthonormal basis***).*

Theorem 5.22 *For a non-finite-dimensional Hilbert space H the following statements are equivalent:*

1. *H is separable.*
2. *All orthonormal bases of H are countable.*
3. *H has a countable basis.*

Proof. (1) $\Rightarrow$ (2): Suppose there is an uncountable orthonormal basis B. As we have $||b-a|| = \sqrt{2}$ for all $a, b \in B, a \neq b$, the unit balls around the elements of B are all disjoint, so there cannot be a countable dense subset of H.
(2) $\Rightarrow$ (3): trivial, as H has an orthonormal basis due to Theorem 5.20.
(3) $\Rightarrow$ (1) was already treated in the section about Schauder bases. $\square$

Theorem 5.23 *If S and T are orthonormal bases of a Hilbert space H, then they have the same cardinality.*

Proof. W.r.o.g. let S be at least countable. Now for every element s of S, the set $T_s := \{t \in T | \langle t, s\rangle \neq 0\}$ is countable due to Theorem 5.17. Therefore and because of $S^\perp = \{0\}$ we get the estimate $|T| \leq |S| \cdot |\mathbb{N}| = |S|$. The same argument applies conversely giving $|S| \leq |T|$. With the set-theoretic Theorem of Schröder-Bernstein follows $|S| = |T|$. $\square$

Theorem 5.24 *If S is an orthonormal base of a Hilbert space H, then H is isometric to $l^2(S)$.*

Proof. Define a linear map $L : H \to l^2(S)$ by $A(v)(s) := \langle v, s\rangle$. This map does take values in $l^2(S)$ because of Bessel's inequality, and Parseval's equality implies that it is an isometry. Conversely, define a linear map $K : l^2(S) \to H$ by $a \to \sum a(s)\cdot s$. Let s_n be an ordering of the points in S where a does not vanish. Then for $N, M \to \infty$,

$$||\sum_{i=N}^{M} a(s_i) \cdot s_i|| = \sum_{i=N}^{M} |a(s_i)|^2 \to 0,$$

therefore the series is Cauchy and converges. With the same argument as in Theorem 5.19 one can show that the limit does not depend on the chosen ordering. Therefore the map K is well-defined, and it is obviously an inverse to L. $\square$

Together with Theorem 5.22, this theorem has the shocking consequence:

Theorem 5.25 *Every infinite-dimensional separable Hilbert space is isometric to $l^2(\mathbb{N})$.* $\square$

If we identify Hilbert spaces by isomorphisms, we have only one infinite-dimensional separable Hilbert space! Considering this poorly diversified zoo, we should be glad that we begun our journey with more general tvs than Hilbert spaces... :-)
Exercise: Give an example of a scaled space X and a linear supspace S of X with $\overline{S} \neq (S^\perp)^\perp$, and one with $\overline{S} \oplus S^\perp \neq X$!
Exercise: Let H be a Hilbert space and $A \in BL(H, H)$. Show that $ker(A) = (A^*(H))^\perp$!
Exercise: Let $B := C^0([a, b], \mathbb{R})$, and let $K \in C^0([a, b]^2, \mathbb{R})$, and consider the operator $A_K : B \to B$ given by $A_K(f)(y) := \int_a^b K(x, y)f(x)dx$. Show that for every real r there is a natural number N only depending on r and K such that if $f_1, ...f_n$ is an L^2-orthonormal system in an eigenspace of A_K to the eigenvalue r, then $n \leq N$. Hint: Use Bessel's inequality!
Exercise for the physicists: Show how by the Gram-Schmidt method the Legendre polynomials as an orthonormal basis of $L^2([-1, 1])$ are constructed out of the basis of the monomials!
Exercise: Let $U \subset \mathbb{R}^n$ be a bounded open subset. Show that for any $f \in L^2(U, \mathbb{R})$ there is a $u \in W^{1,2}(U, \mathbb{R})$ satisfying $-\Delta u = f$ in U and $u = 0$ in ∂U. **Hint:** Use partial integration and introduce a scalar product $\langle g, h\rangle' := \langle g, h\rangle_{W^{1,2}} - \langle g, h\rangle_{L^2}$ on $W_0^{1,2}$. Finally, to apply the THeorem of Fréchet-Riesz, show the equivalence of $\langle\cdot,\cdot\rangle'$ and $\langle\cdot,\cdot\rangle_{W^{1,2}}$ using the following

Theorem 5.26 (Poincaré-Friedrichs inequality) *Let $U \subset \mathbb{R}^n$ be open and bounded, then, for $s := diam(U)$,*

$$||u||L^2 \leq 2s\langle u, u\rangle'$$

for all $u \in W_0^{1,2}(U)$ (the limits of functions vanishing on the boundary).

Proof. Both sides of the equation depend continuously on the $W^{1,2}$-norm, so because of density of C^1 in its completion it is sufficient to show the inequality for $u \in C_0^1(U, \mathbb{R})$. Now choose $[-s, s]^n \supset U$ (w.r.o.g. byy translation of U) and extend u to 0 outside U, then for $x = (x_1, ...x_n) \in [-s, s]^n$ we get

$$u(x) = \int_{-s}^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)dt = \int_{-s}^{s} \chi_{[-s,x_1]}(t)\frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)dt$$

where ξ_I is the characteristic function of the interval I. The Cauchy-Schwary inequality implies

$$|u(x)|^2 \leq \int_{-s}^{s} (\xi_{[-s,x_1]}(t))^2 dt \cdot \int_{-s}^{s} |\frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)|^2 dt \leq 2s \int_{-s}^{s} |\frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)|^2 dt,$$

and therefore

$$\begin{aligned}
\int_U |u(x)|^2 &= \int_{-s}^{s} dx_1 ... \int_{-s}^{s} dx_n |u(x)|^2 \\
&\leq 2s \int_{-s}^{s} dx_1 ... \int_{-s}^{s} dx_n \int_{-s}^{s} |\frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)|^2 dt \\
&= (2s)^2 \int_{-s}^{s} dx_2 ... \int_{-s}^{s} dx_n \int_{-s}^{s} dt |\frac{\partial u}{\partial x_1}(t, x_2, ..., x_n)|^2 \\
&= 4s^2 \int_U |\frac{\partial u}{\partial x_1}(x)| dx \leq 4s^2 \langle u, u\rangle'
\end{aligned}$$

which is the required statement. □